On DOCTOR HALLEY'S SERIES *for the Calculation of* LOGARITHMS, *by the* REV. RICHARD MURRAY, *D. D. late* PROVOST *of Trinity College near Dublin.**—*Read Nov. 16th,* 1801.

LORD NEPER was, without diſpute, the inventor of logarithms. In the year 1614 he publiſhed a ſketch of his plan, together with ſome ſhort tables of logarithms calculated by himſelf, under the title of mirifici logarithmorum canonis deſcriptio; and the obvious uſes of the ſcheme were ſo many and great, that the invention was received with eagerneſs by almoſt all the mathematicians in Europe, many of whom ſet themſelves about improving the hint, and calculating larger tables. Of theſe the principal was Mr. Henry Briggs, who went twice into Scotland to conſult with the inventor; at which times they agreed upon ſome alterations to be made in the tables, and ſettled the plan of thoſe logarithms that are the moſt convenient of all for uſe, and which are known by the name of Vulgar Logarithms; they are alſo frequently called Briggs's Logarithms, becauſe, Lord Neper dying ſhortly after their ſecond conſultation, the whole buſineſs of forming the tables devolved upon Briggs.

 Theſe

* This eſſay was found among the papers of Doctor Murray, after his death. He had drawn it up, ſhortly after his appointment in 1763 to the profeſſorſhip of mathematics, for the inſtruction of his pupils, and much of it therefore is employed in explanations, which, had he deſigned it for publication in the preſent mode, he would have retrenched; but, as the whole is ſhort, it has been thought adviſeable to give it in its original form.

Thefe two (Neper and Briggs, whom we may call the firft authors of logarithms) ufed two methods in their computations: the one by involution, or raifing the number whofe logarithm they fought to a certain power; the other by evolution, or extracting out of it a root the denominator of whofe index was fufficiently great; and this latter they were obliged to do by repeated extractions of the fquare root, no eafier method being then known.

But afterwards, when Sir Ifaac Newton's famous binomial theorem was made public, Doctor Halley took advantage of that invention, and fhewed a method of calculating logarithms by throwing the root required (or rather the logarithm derived from it) into a converging feries: and this method is as eafy and expeditious as can ever be expected, or indeed defired, the law of the feries being obvious, the terms eafily reduced to numbers, and a very few of them fufficient.

This difference in the manner of extracting the root makes the principal difference between the methods of finding the logarithms ufed by the firft authors and by Halley.

There are fome properties of thefe roots which are neceffary to be known, and which are obvious enough when the roots are found out in Briggs's way, but which require proof when they are found by the binomial theorem. Thefe properties are here premifed in the form of Lemma's, that the explanation of Halley's method may not be interrupted by proving them afterwards.

Lemma. 1.

"Let c be a proper fraction, and n any whole number confifting of many places of figures; if out of the binomial $1+c$ a root is to be extracted whofe index is $\frac{1}{n}$, it is required to find the moft fimple feries that fhall give that root true to a number of places of decimals lefs by four or more than twice the number of places of figures in n."

$$\text{(A) } 1.\quad \frac{1}{n}e.\quad \frac{\frac{1}{n}-1}{2}e\quad \frac{\frac{1}{n}-2}{3}e\quad \frac{\frac{1}{n}-3}{4}e\quad \frac{\frac{1}{n}-4}{5}e\ \text{\&c.}$$

$$\text{(B) } 1.\quad \frac{1}{n}e.\quad \frac{-1}{2}e.\quad \frac{-2}{3}e.\quad \frac{-3}{4}e.\quad \frac{-4}{5}e.\quad \text{\&c.}$$

$$\text{(C) } 1. + \frac{1}{n}e. - \frac{1}{2n}e.^2 + \frac{1}{3n}e.^3 - \frac{1}{4n}e.^4 + \frac{1}{5n}e.^5\ \text{\&c.}$$

By multiplying together the terms of the firſt ſeries above marked A, according to the method of the binomial theorem, a new ſeries will be produced, which will give that root true to any number of places of decimals: but in the third and following terms of the ſeries A, the fraction $\frac{1}{n}$ is ſo ſmall, that it may be omitted without cauſing any error within the preſcribed number of places, and this will reduce the ſeries A, to the ſecond ſeries above marked B, and the terms of it, multiplied according to the method of the binomial theorem, produce the third ſeries marked C, which therefore is the ſeries required. In it the firſt term is 1, and the reſt are formed by the following laws.

1ſt, In the 2d, 3d, 4th, and following terms are the ſeveral powers of e, whoſe indices are 1, 2, 3, &c. the natural numbers.

2dly, The co-efficient or uncia of any term is $\frac{1}{n}$ divided by the index of e in that ſame term: for it is $\frac{1}{n}$ divided by the denominator of the laſt of the fractions in the ſeries B, that were multiplied together in order to produce that term.

3dly, The two firſt terms of it are poſitive, becauſe the two firſt terms in B are poſitive: but the following terms are alternately negative and poſitive; becauſe the 3d and following terms of B being all negative, in the 3d and following terms of C there will be alternately an odd and even number of negative factors.

And

And that the omiſſion of the fraction $\frac{1}{n}$ in the third and following terms of the ſeries A will not cauſe any error within the preſcribed numbers of places of figures, will appear by conſidering the value of the quantity that is thereby omitted in any term of the ſeries C: for it will appear for many reaſons to be always leſs than $\frac{1}{nn}$, the greateſt poſſible value of which is a decimal conſiſting of 1 preceded by twice as many cyphers, except 3, as there are places in n; and therefore the value of all the omiſſions in 100 terms will not amount to an unit in the laſt of the preſcribed places.

Lemma 2.

"The values of c and n continuing as before, it is required to find a like ſeries that ſhall give the root of the reſidual $1-e$ true to the ſame number of places."

$$\text{(D)}\quad 1. \quad \frac{1}{n}\times -c. \quad \frac{-1}{2}+ -c. \quad \frac{-2}{3}\times -e. \quad \frac{-3}{4}\times -e. \quad \frac{-4}{5}\times -e. \ \&c.$$

$$\text{(E)}\quad 1 - \frac{1}{n}e - \frac{1}{2n}e^2 - \frac{1}{3n}e^3 - \frac{1}{4n}e^4 - \frac{1}{5n}e^5 \quad \&c.$$

In every term after the firſt of the ſeries B change the ſign of the quantity c, and there will reſult the ſeries marked D, whoſe terms, multiplied as before, produce the ſeries required, which is marked E. This ſeries differs from C only in this, that all the terms after the firſt are negative: for in the ſecond term of D, e is negative; therefore the ſecond term of E (having one negative factor) is negative; and in every ſucceeding term of D, there are two negative factors, whence in every following term of E, there will be an odd number of negative factors, and therefore they will all be negative.

Lemma 3.

“ The values of e and n continuing as before, let m be any whole number different from n, but like it consisting of many places of figures; if out of the binomial or residual $1 \pm e$ be extracted roots whose indices are $\frac{1}{n}$ and $\frac{1}{m}$, and if the roots be calculated true to numbers of places of decimals less by four or more than twice the numbers of places of figures in either n or m; and lastly if from each root be subtracted unity, then will the remainders be to each other as the indices $\frac{1}{n}$ and $\frac{1}{m}$, or reciprocally as the denominators n and m.”

For if 1 be subtracted from the root whose index is $\frac{1}{n}$, by the two preceding lemmas, the remainder will be $\pm \frac{1}{n}e - \frac{1}{2n}e^2 \pm \frac{1}{3n}e^3 - \frac{1}{4n}e^4 \pm \frac{1}{5n}e^5$ &c. which is equal to $\frac{1}{n} \times \overline{\pm e - \frac{1}{2}e^2 \pm \frac{1}{3}e^3 - \frac{1}{4}e^4 \pm \frac{1}{5}e^5}$ &c. and in like manner, if 1 be subtracted from the root whose index is $\frac{1}{m}$, the remainder will be $\frac{1}{m} \times \overline{\pm e - \frac{1}{2}e^2 \pm \frac{1}{3}e^3 - \frac{1}{4}e^4 \pm \frac{1}{5}e^5}$ &c. and it is evident that these series are to each other as $\frac{1}{n}$ and $\frac{1}{m}$, or reciprocally as n and m.

Lemma 4.

“ The values of e and n continuing as before, if out of the quantity $1 \pm e$ a root is to be extracted whose index is $\frac{1}{n}$; and unity being subtracted from the root, if the remainder is to be multiplied by the denominator n; it is required to find the most simple series that shall give the product true to a number of places of decimals, less by three or more than the number of places of figures in n.”

From what is said in the first and second lemmas it is plain that the series required is $\pm e - \frac{1}{2}e^2 \pm \frac{1}{3}e^3 - \frac{1}{4}e^4 \pm \frac{1}{5}e^5$ &c. the laws of whose continuation

continuation are as follow: firſt, the indices of e in the ſeveral terms are the natural numbers in their order: ſecondly, the numeral co-efficient of any term is 1 divided by the index of e in that term: and thirdly, when the quantity is $1 + e$, the terms are alternately affirmative and negative; but when it is $1 - e$, the terms are all negative.

And that the omiſſion of the fraction $\frac{1}{n}$ in the third and following terms of the ſeries A (in Lemma 1) will not cauſe any error here, within the preſcribed number of places, will appear by conſidering the value of the quantity that is omitted in any term of the ſeries of this Lemma: for it appears to be always leſs than $\frac{1}{n}$, the greateſt poſſible value of which is a decimal conſiſting of 1 preceded by as many cyphers except two as there are places of figures in n; and therefore the omiſſions in 100 terms will not amount to an unit in the laſt of the preſcribed places.

Note, That Doctor Halley has derived the ſubſtance of theſe lemmas in a ſhorter manner, from a ſuppoſition of n and m, the denominators of the indices, being infinite numbers. But as there may be ſome difficulty in conceiving this, I have propoſed as much of the doctrine as was wanted here, without making that ſuppoſition.

Art. 1. By ratiuncula I underſtand a ratio of inequality, but which is very near to a ratio of equality. Thus if r be a decimal fraction having many cyphers (as 10, 20, 50, 100) before the firſt ſignificant figure, the ratio of 1 to $1 + r$, or of 1 to $1 - r$ may be called a ratiuncula.

2. Logarithims are the exponents (or numeral meaſures) of ratios.

3. Now ratios are meaſured by the numbers of equal ratiunculæ of which they are compounded, or into which they may be ſuppoſed to be reſolved. Thus, if between 1 and 10 be placed 999999999 mean proportionals, then will the ratio of 1 to 10 be reſolved into 1000000000 ratiunculæ, each equal to the ratio of 1 to the firſt or leaſt of thoſe means: and if of theſe means 301029995 ſtand between 1 and 2 the

the ratio of 1 to 2 will be refolved into 301029995 of thefe ratiunculæ; and thefe numbers, 1000000000 and 301029995, will be the logarithms of the ratios of 1 to 10 and of 1 to 2, as being the numbers of equal ratiunculæ of which thefe ratios are compounded.

4. But though thefe numbers be immediately and properly the logarithms of thefe ratios, they are not the only ones that can be ufed as fuch; any two numbers (or indeed any two quantities of the fame kind) that have the fame ratio with them, may be made their logarithms. Thus, if there be any convenience in having 1 for the logarithm of the ratio of 1 to 10, and if 0,301029995 be to 1, as 301029995 is to 1000000000, then may 1 and 0,301029995 be made the logarithms of thofe ratios.

5. If 1 be made the antecedent of any ratio, that ratio may be refolved into any number of equal ratiunculæ, by extracting out of the confequent a root, the dominator of whofe index is the number of ratiunculæ that is required. Thus, if it be required to refolve the ratio of 1 to $1+e$ into n ratiunculæ, the ratio of 1 to $\overline{1+e}^{\frac{1}{n}}$ will be the firft of them; and it is fufficient to find one of them, each of the others being equal to it.

6. Let now e and y be any two numbers, of which e is the greater, and between 1 and $1+e$ let a feries of mean proportionals be placed, whofe number is $n-1$, the ratio of 1 to $1+e$ will be refolved into n ratiunculæ; if of thefe means any number denoted by m ftands between 1 and $1+y$, the ratio of 1 to $1+y$ will be refolved into m ratiunculæ, each equal to one of the former; and from what was faid above (art. 3.) the logarithms of the ratio's of 1 to $1+e$ and of 1 to $1+y$ will be as n and m.

7. The firft of thefe mean proportionals is $\overline{1+e}^{\frac{1}{n}}$, and if this quantity be involved to a power whofe index is m, that power will be equal to $1+y$, (or fo near to it, that it may be ufed for it without any error,) that is $\overline{1+e}^{\frac{m}{n}} = 1+y$, and therefore $\overline{1+e}^{\frac{1}{n}} = \overline{1+y}^{\frac{1}{m}}$. Suppofe now that

out of $1+y$ is extracted a root whose index is $\frac{1}{n}$, that root will be $\overline{1+y}^{\frac{1}{n}}$; and since (by lem. 3.) $\overline{1+y}^{\frac{1}{m}}-1 : \overline{1+y}^{\frac{1}{n}}-1 :: n : m$, it follows that $\overline{1+e}^{\frac{1}{n}}-1 : \overline{1+y}^{\frac{1}{n}}-1 :: n : m$. But it was proved above (art. 6.) that the logarithms of the ratios of 1 to $1+e$ and of 1 to $1+y$, are as n and m; therefore these logarithms are as $\overline{1+e}^{\frac{1}{n}}-1$ and $\overline{1+y}^{\frac{1}{n}}-1$; that is, if out of two numbers both greater than 1 be extracted the same root, the excesses of these roots above 1 will be as the logarithms of the ratios that 1 has to these numbers: and therefore if one of these excesses $\overline{1+e}^{\frac{1}{n}}-1$ (or any multiple of it) be made the logarithm of the ratio of 1 to $1+e$, the other excess $\overline{1+y}^{\frac{1}{n}}-1$ (or the same multiple of it) must be made the logarithm of the ratio of 1 to $1+y$.

8. Now according to Neper's first plan, as published in the Canon Mirificus, when he resolved any ratio (as of 1 to $1+e$) into a sufficient number of ratiunculæ, or which is the same thing, when he had placed between 1 and $1+e$ a sufficient number of mean proportionals, he made the excess of the first, or least of them above 1, to be the logarithm of one of these ratiunculæ: thus, if $\overline{1+e}^{\frac{1}{n}}$ be the first, or least, of the mean proportionals between 1 and $1+e$, then he made $\overline{1+e}^{\frac{1}{n}}-1$, to be the logarithm of the ratio of 1 to $\overline{1+e}^{\frac{1}{n}}$, or, as it is usually called, the logarithm of the number $\overline{1+e}^{\frac{1}{n}}$; and then it will follow from the nature of logarithms, that $n\times\overline{\overline{1+e}^{\frac{1}{n}}-1}$ will be the logarithm of the ratio of 1 to $1+e$, or of the number $1+e$; and hence again (by what was said in art. 7.) it follows, that $n\times\overline{\overline{1+y}^{\frac{1}{n}}-1}$ must be the logarithm of the ratio of 1 to $1+y$, or of the number $1+y$.

9. Let

9. Let now e and y be proper fractions, and (by lemma 4) $n \times \overline{1+e}^{\frac{1}{n}} - 1$ $= e - \frac{1}{2}e^2 + \frac{1}{3}e^3 - \frac{1}{4}e^4 + \frac{1}{5}e^5$ &c. and also $n \times \overline{1+y}^{\frac{1}{n}} - 1 = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \frac{1}{5}y^5$ &c.; and thus by series of this kind we can find Neper's logarithms of all mixed numbers between 1 and 2.

10. In like manner, if the number whose logarithm is sought, be less than 1, that is, if the ratio be that of 1 to $1-e$, having placed a sufficient number of mean proportionals between 1 and $1-e$, he subtracted 1 from $\overline{1-e}^{\frac{1}{n}}$, the first or greatest of them, and made the remainder $\overline{1-e}^{\frac{1}{n}} - 1$ (which now becomes negative) to be the logarithm of one ratiuncula, and therefore $n \times \overline{1-e}^{\frac{1}{n}} - 1$ will be the logarithm of the ratio of 1 to $1-e$, or of the number $1-e$; and hence as before, $n \times \overline{1-y}^{\frac{1}{n}} - 1$ must be the logarithm of the ratio of 1 to $1-y$, or of the number $1-y$. But (by lemma 4) $n \times \overline{1-e}^{\frac{1}{n}} - 1 = -e - \frac{1}{2}e^2 - \frac{1}{3}e^3 - \frac{1}{4}e^4 - \frac{1}{5}e^5$ &c., and also $n \times \overline{1-y}^{\frac{1}{n}} - 1$ $= -y - \frac{1}{2}y^2 - \frac{1}{3}y^3 - \frac{1}{4}y^4 - \frac{1}{5}y^5$ &c., and therefore, by series of this kind, we can find the logarithms of all numbers less than 1.

11. Let now any two numbers be proposed, a the lesser, and b the greater, the logarithm of whose ratio is required. We must first find a ratio whose antecedent is 1, and which shall be equal to the ratio of a to b: this is done by finding the value of e in the following analogy, $a : b :: 1 : 1+e$; which being changed into an equation, becomes $a + ae$ $= b$, whence $ae = b - a$, and therefore $e = \frac{b-a}{a}$: and if e be a proper fraction, we may then find the hyperbolic logarithm of the ratio of 1 to $1+e$, which (by art. 9.) appears to be $e - \frac{1}{2}e^2 + \frac{1}{3}e^3 - \frac{1}{4}e^4 + \frac{1}{5}e^5$ &c.; and since equal ratios have the same logarithm, that series will also be the logarithm of the ratio of a to b.

12. And if the ratio be that of b to a, we must still find an equal ratio whose antecedent is 1, which is done by finding the value of e in this

this analogy $b : a :: 1 : 1-e$, which gives this equation $b - be = a$, whence $be = b-a$, and $e = \frac{b-a}{b}$: here e muſt neceſſarily be a proper fraction, and the logarithm of the ratio of 1 to $1-e$ (by art. 10.) is $-e-\frac{1}{2}e^2-\frac{1}{3}e^3-\frac{1}{4}e^4-\frac{1}{5}e^5$ &c., which is therefore the logarithm alſo of the ratio of b to a.

13. From art. 11, we may obſerve that when the given ratio is aſcending, or of leſſer inequality, the value of e is the difference of the given terms divided by the leſſer of them: and from art. 12, that when the given ratio is deſcending, or of greater inequality, the value of e is the difference of the ſame terms divided by the greater.

14. Either of the above ſeries might be ſufficient for finding all logarithms; but by joining the two together a third ſeries reſults, much more convenient for the purpoſe, as it converges twice as faſt as either of them; the method of doing it (which muſt be carefully attended to) is as follows.

15. Between a and b, the terms of the given ratio, place p an arithmetical mean; the whole ratio of a to b is thereby reſolved into two, that of a to p, and of p to b: invert the former, and it becomes the ratio of p to a; and if we make $1 : 1-e :: p : a$, then (by art. 13) will $e = \frac{p-a}{p}$, and the logarithm of the ratio of 1 to $1-e$, or of p to a, is the ſeries $-e-\frac{1}{2}e^2-\frac{1}{3}e^3-\frac{1}{4}e^4-\frac{1}{5}e^5$ &c. by art. 10.

16. Again, if we make $1 : 1+e :: p : b$, e will be $\frac{b-p}{p}$ by art. 13, but, ſince p is an arithmetical mean between a and b, $\frac{b-p}{p} = \frac{p-a}{p}$; therefore e has the ſame value as in the laſt article; and the logarithm of the ratio of 1 to $1+e$, or of p to b, is $e-\frac{1}{2}e^2+\frac{1}{3}e^3-\frac{1}{4}e^4+\frac{1}{5}e^5$ &c. by art. 9.

17. In art. 15, the logarithm of the ratio of p to a was found to be $-e-\frac{1}{2}e^2-\frac{1}{3}e^3-\frac{1}{4}e^4-\frac{1}{5}e^5$ &c. Invert this ratio again, and it becomes the ratio of a to p, and its logarithm is the ſame as before, only its ſign is changed: that is, the logarithm of the ratio of a to p is the ſeries $e+\frac{1}{2}e^2+\frac{1}{3}e^3+\frac{1}{4}e^4+\frac{1}{5}e^5$ &c., and by art. 16, the logarithm of the ratio of p to b is $e-\frac{1}{2}e^2+\frac{1}{3}e^3-\frac{1}{4}e^4+\frac{1}{5}e^5$ &c., and therefore the logarithm of the compound ratio, or of a to b, is the ſum of theſe two ſeries, which is $2e+2\times\frac{1}{3}e^3+2\times\frac{1}{5}e^5$ &c., or $2\times\overline{e+\frac{1}{3}e^3+\frac{1}{5}e^5 \text{ \&c.}}$

18. In

18. In art. 15 e was found to be $\frac{p-a}{p}$, and in art. 18 e was found to be $\frac{b-p}{p}$; and it was there obferved, that $\frac{p-a}{p} = \frac{b-p}{p}$, becaufe p is an arithmetical mean between a and b; from which it alfo follows that either numerator, $p-a$ or $b-p$, is equal to $\frac{1}{2}b-\frac{1}{2}a$, and alfo that the common denominator p is equal to $\frac{1}{2}b+\frac{1}{2}a$, therefore e is always equal to $\frac{\frac{1}{2}b-\frac{1}{2}a}{\frac{1}{2}b+\frac{1}{2}a}$, or to $\frac{b-a}{b+a}$; that is, the value of e is always a fraction whofe numerator is the difference of the terms of the given ratio, and whofe denominator is their fum: and the logarithm of the ratio is the double of a feries formed by the following laws: 1ft. the feveral terms of the feries contain the powers of that fraction or quantity whofe indices are the odd numbers; 2dly, every term is divided by the index of the power of the quantity e in it; and 3dly, the terms are all affirmative, when the ratio is that of a to b, or afcending; but would all be negative if the ratio were that of b to a, or defcending. And by thefe feries may be found the logarithms that are called Neper's Logarithms, and fometimes the Natural Logarithms, but moft ufually the Hyperbolic Logarithms of Numbers or Ratios.

This is Doctor Halley's method, as far as it relates to logarithms in general. But it may be neceffary to add fome obfervations upon it, and particularly to affign the reafons of the feveral operations where thefe reafons are not fufficiently obvious of themfelves.

1. I have throughout fuppofed that the logarithms of an afcending ratio (or of leffer inequality) is affimative, and that the logarithm of a defcending ratio (or of greater inequality) is negative; but this is a matter in its own nature abfolutely indifferent: the logarithms of a ratio of either inequality may be made affirmative; but then the logarithms of ratios of the other inequality muft be negative, and reciprocal ratios muft have logarithms equal in quantity, but with unlike figns.

2. In art. 11 and 12, I have fuppofed every ratio to be fo reduced, as that its antecedent may be 1. I might have reduced them fo as to make 1 the confequent of each. But the neceffity of one or other will appear from hence, that the logarithms of ratios are found by in-

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ſerting ſufficient numbers of mean proportionals between their terms: for if between two numbers, a and b, it be required to inſert a ſeries of mean proportionals whoſe number is m, the firſt of them will be $\overline{a^m b}|^{\frac{1}{m+1}}$, and the laſt will be $\overline{a b^m}|^{\frac{1}{m+1}}$; in each caſe one of the given terms muſt be involved to a power whoſe index is the number of means required; and this, we may ſafely ſay, would be impracticable in the preſent caſe, if that term be different from 1, on account of the greatneſs of the number m: but if that term be 1, this trouble is wholly avoided, every power of 1 being ſtill 1.

Again, the other term of every ratio is propoſed under this form $1 \pm e$, that is as a binomial or reſidual of which the firſt member is 1; the reaſon of which will appear from this, that if the m^{th} power of a binomial or reſidual $a \pm b$, be to be found by the binomial theorem, the firſt, ſecond, third, and following terms of the ſeries will contain the powers of a whoſe indices are m, $m-1$, $m-2$ &c. that is a^m will be the firſt term, a^{m-1} will be one factor of the ſecond term, a^{m-2} one of the third term, and ſo on; and if (in the caſe of calculating logarithms) a be different from 1, it may ſafely be pronounced impracticable to find thoſe powers of a: whereas if a be made equal to 1, all that trouble vaniſhes, every power of 1 being ſtill 1.

And laſtly, e, the ſecond member of the binomial or reſidual, is ſuppoſed to be a proper fraction; for otherwiſe the ſeries of art. 9, would either perpetually diverge, or after converging ſlowly for ſome time, would afterwards diverge; or laſtly, would converge perpetually, but ſo ſlowly as to be totally uſeleſs: but we need not inſiſt further upon thoſe particulars, becauſe in the ſeries of art. 17, which is the only one that we can ever have occaſion to reduce to numbers, the quantity e muſt always be a proper fraction, its numerator being the difference, and its denominator the ſum of the terms of the given ratio. Some uſeful cautions however may be given, relating to that fraction; as that it is convenient to have 1 for its numerator; for each

ſucceeding

fucceeding term of the feries is to be derived from the preceding, and if its numerator be not 1, there will be a neceffity of both multiplying and dividing; whereas by making its numerator 1, the multiplication is avoided. And for this reafon, if a ratio be propofed whofe terms will not immediately give a fraction of this kind, it is to be refolved into others, in each of which the difference of the terms is either 1, or a meafure of their fum. Thus, if the logarithm of the ratio of 5 to 8, were required, from what has hitherto been explained, the quantity e would be $\frac{3}{13}$; but, inftead of immediately finding that logarithm, the ratio is to be refolved, either into the ratios of 5 to 6 and of 6 to 8 and then the fractions become $\frac{1}{11}$ and $\frac{1}{7}$; or into the ratios of 5 to 7 and of 7 to 8, and then the fractions become $\frac{1}{6}$ and $\frac{1}{15}$; or laftly into the ratios of 5 to 6, and of 6 to 7, and of 7 to 8, and the fractions become $\frac{1}{11}$, $\frac{1}{13}$, and $\frac{1}{15}$: and the logarithms of any of thefe fets of ratios being found, their fum will be the logarithm of the ratio of 5 to 8.

There is alfo frequently another reafon for refolving the ratio firft propofed, into others; and that is in order to diminifh the fraction e; for as it is diminifhed, the feries converges the fafter, and it may frequently be eligible to find two or three or more logarithms by feries that converge faft, rather than one by a feries that converges flowly.

3. In art. 15, by adding two feries together, a third feries refults more fimple than either of them. The feveral fteps, by which this is effected, are now to be explained.

After the given ratio is refolved into two, it is ordered that one (and one only) of thefe ratios be inverted; for if neither of them, or both, were inverted, they would ftill be, either both afcending, or both defcending; and, in either cafe, the two feries produced would have their correfpondent terms (i. e. terms that involve the fame power of the literal quantity e) affected by like figns, and therefore no term would vanifh by addition. Whereas by inverting one ratio only, one feries has all its terms affected by the fame fign, and the other has its terms

alternately

alternately affirmative and negative; and therefore the two series have their alternate correspondent terms affected by unlike signs; these terms therefore may vanish by addition. But in order to this, it is also necessary that the quantities (both numeral and literal) of these terms should be the same. Now the numeral quantity, or coefficient, must be the same in correspondent terms of these series, because each is 1 divided by the index of the literal quantity e in that term; it only remains therefore that care be taken to have the quantity e of the same value in both series; and this is done by providing that both its numerator and denominator be the same: and that its numerator is the same, follows from p, the quantity inserted between a and b, being an arithmetical mean between them; for $p - a$ and $b - p$ are the numerators; and that the denominator is the same, follows from its being the first of the two ratios that is inverted, for p, the quantity inserted, must always be the denominator of both fractions. This appears, when the given ratio is ascending, from what was said in art. 15 and 16: and if the given ratio had been descending, as of b to a, still it is to be resolved into the ratios of b to p and of p to a; and if the first of them be inverted, it becomes the ratio of p to b, or ascending, and therefore by art. 13, p will be the denominator of the fraction: and the other ratio, that of p to a, being still descending, by art. 13, p will be the denominator of the fraction here also; and thus the second terms of the two series, and the alternate terms from them, being composed of the same quantities, both literal and numeral, and having unlike signs, they will entirely vanish when the series are added together.

It may be proper here to observe, that the two rules, (that for making p, the inserted term, an arithmetic mean between a and b; and that for inverting the former of the two ratios,) become necessary together; that is, they are so connected together, as that when either is observed, the other must be observed also. But we may neglect both these rules, and yet arrive at the same conclusion, by the following rules: divide the difference of the given terms into two parts proportional

tional to theſe terms, and to the leſſer term add the leſſer of theſe parts, and make that ſum the intermediate term, and then invert the latter of the two ratios. But ſince this method has no advantage over the other, and ſince the proof of it is not ſo obvious, Doctor Halley juſtly paſſed it over in ſilence.

The doctrine delivered in art. 7. may perhaps become clearer by being divided into ſeveral propoſitions, as follows.

PROP. 1.

The logarithms of two different powers of the ſame number are to each other as the indices of the powers.

For the logarithms of theſe powers are the products of the logarithm of the root into the reſpective indices; and therefore are to each other as the indices.

PROP. 2

If out of two numbers be extracted roots, whoſe indices are ſuch that the roots themſelves may be equal, the logarithms of thoſe numbers will be to each other as the denominators of the indices of the roots.

For if the common root be raiſed to a power, whoſe index is the greater denominator, that power will be the greater number; and if the ſame root be raiſed to a power whoſe index is the leſſer denominator, that power will be the leſſer number; and therefore (by the preceding prop.) the logarithms of the numbers will be as the denominators.

PROP. 3.

If out of a number, which ſtands between 1 and 2, be extracted different roots, the denominators of whoſe indices are

numbers consisting of many places of figures, the excesses of these roots above unity will be to each other as the indices of the roots, or reciprocally as the denominators of the indices; provided that the roots be calculated only to a number of places of figures less by 2 or 3 than twice the number of places in the lesser denominator.

This proposition is the same with lemma 3, and has been proved before.

PROP. 4.

If out of two numbers, both standing between 1 and 2, be extracted roots, whose indices are the same, and of which the denominator is sufficiently great, the excesses of the roots above unity will be to each other as the logarithms of the numbers themselves.

For suppose 1st, that roots are extracted out of the numbers whose indices are such that the roots themselves may be equal; then (by prop. 2d) the logarithm of the greater number will be to the logarithm of the lesser as the greater denominator is to the lesser. Suppose 2dly, that out of the lesser number another root is extracted, whose index is the same with the index of the root extracted out of the greater number; there are now extracted out of the lesser number two different roots, and (by prop. 3.) the excesses of the greater and lesser of these roots above unity will be to each other as the greater and lesser denominator; that is (as was proved above) as the logarithms of the greater and lesser number. But the greater of these roots is equal (by supposition) to the root extracted out of the greater number; therefore the excesses above unity of the roots extracted out of the greater and lesser number when the index is the same, are to each other as the logarithms of the numbers themselves.

The same argument in symbols is in the next page.

Let

Let e and y be proper fractions of which e is the greater; then will $1 + e$ be the greater number, and $1 + y$ the lesser: also let n be the greater denominator, and m the lesser:

By supposition -	1	$\overline{1+e}^{\frac{1}{n}} = \overline{1+y}^{\frac{1}{m}}$
By prop. 2nd. -	2	Log. of $1 + e$: Log. of $1 + y :: n : m$.
By lemma 3d. -	3	$\overline{1+y}^{\frac{1}{m}} - 1 : \overline{1+y}^{\frac{1}{n}} - 1 :: n : m$.
2 + 3 -	4	$\overline{1+y}^{\frac{1}{m}} - 1 : \overline{1+y}^{\frac{1}{n}} - 1 ::$ Log. of $1 + e$: Log. of $1 + y$
1 + 4 -	5	$\overline{1+e}^{\frac{1}{n}} - 1 : \overline{1+y}^{\frac{1}{n}} - 1 ::$ Log. of $1 + e$: Log. of $1 + y$